On the enumeration of Krom functions

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Arguments (3-bitvectors), sorted by decreasing lexicographic order →

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Fact (enumeration of Boolean functions). There are 2^{2^n} possible n-ary Boolean functions.

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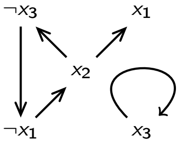

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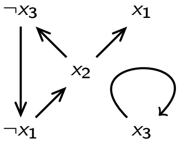

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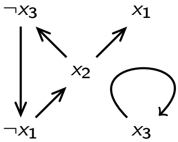

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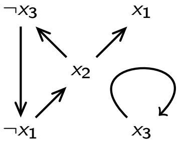

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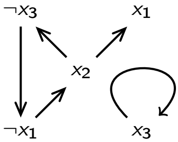

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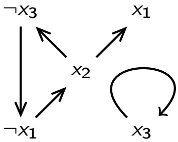

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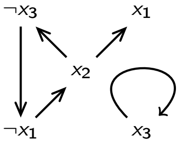

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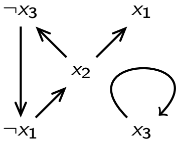

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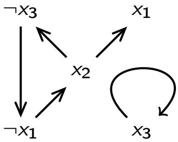

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Note. Only the first eight terms of K are currently known: 4, 16, 166, 4170, 224716, 24445368, 5167757614, 2061662323954 (computed by Knuth, see OEIS A109457).

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Note. There are palindromic truth vectors which do not represent any Krom function, for example 01111110.