On the enumeration of Krom functions

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Dutch Days of Combinatorics, Utrecht, 2023

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A **Boolean function** is an operation on $\mathbb{Z}_2$.

Example. A ternary Boolean function.

Arguments (3-bitvectors), sorted by decreasing lexicographic order →

x_1	x_2	x_3	$(x_1+1)x_3+1$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

← *truth vector* (2^3-bitvector)

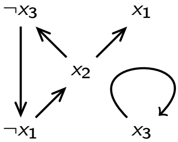
Fact. Every n-ary Boolean function can be represented as a polynomial of $\mathbb{Z}_2[x_1, \ldots, x_n]$, its **Zhegalkin polynomial**.

Fact. Every n-ary Boolean function can be represented as a 2^n-bitvector, its **truth vector**.

Fact (enumeration of Boolean functions). There are 2^{2^n} possible n-ary Boolean functions.

Krom functions

Now consider a **Krom digraph**: a digraph whose vertices are variables or negated variables, e.g. the following one.



A **Krom function** is constructed as follows.

1. List the edges as factors of a product (the order does not matter): $(\neg x_3 \to \neg x_1)(x_2 \to \neg x_3)(x_2 \to x_1)(\neg x_1 \to x_2)(x_3 \to x_3)$.
2. Replace the edges $u \to v$ with $u + uv + 1$: $(\neg x_3 + \neg x_3 \neg x_1 + 1)(x_2 + x_2 \neg x_3 + 1)(x_2 + x_2 x_1 + 1)(\neg x_1 + \neg x_1 x_2 + 1)(x_3 + x_3 x_3 + 1)$.
3. Replace the negations $\neg x_i$ with $(x_i + 1)$: $((x_3+1)+(x_3+1)(x_1+1)+1)(x_2+x_2(x_3+1)+1)(x_2+x_2x_1+1)((x_1+1)+(x_1+1)x_2+1)(x_3+x_3x_3+1)$.
4. Simplify (in $\mathbb{Z}_2[x_1, \ldots, x_n]$): $x_1x_2x_3 + x_1x_3$.

Note. The vast majority of Krom digraphs lead to the zero polynomial.

Open problem (enumeration of Krom functions). What is the number $K(n)$ of possible n-ary Krom functions?

Note. Only the first eight terms of K are currently known: 4, 16, 166, 4170, 224716, 24445368, 5167757614, 2061662323954 (computed by Knuth, see OEIS A109457).

A **Krom graph** is a Krom digraph which is also a graph (i.e. a symmetric Krom digraph or, in other words, a Krom digraph which has double arrows only).

Problem. How many possible Krom functions can we generate if we only input Krom graphs?

Observation. The answer to the above problem is $1 + \mathcal{B}_n(2^0, \ldots, 2^{n-1})$, where $\mathcal{B}_n$ is the **complete Bell polynomial** in n variables,

$$\begin{vmatrix} x_1/0! & \cdots & \cdots & \cdots & x_n/(n-1)! \\ -1 & x_1/0! & \cdots & \cdots & x_{n-1}/(n-2)! \\ & -2 & x_1/0! & \cdots & x_{n-2}/(n-3)! \\ & & \ddots & \ddots & \vdots \\ & & & -(n-1) & x_1/0! \end{vmatrix}.$$

Observation. The above number is also the number of possible n-ary Krom functions whose truth vector is palindromic.

Note. There are palindromic truth vectors which do not represent any Krom function, for example 01111110.